Pat O'Sullivan

Mh4714 Week 5

Week 5

0.0.0.1 Real numbers v. Rational numbers. The Completeness Axiom implies that $\mathbb{R}$ must contain an element whose square is 2, that is, $\mathbb{R}$ contains the number we refer to as $\sqrt{2}$.

There is a real number L such that $L^2 = 2$. That is, $\sqrt{2}$ exists in $\mathbb{R}$.

To prove this, we will construct an infinite decimal which converges to a real number L with the property that $L^2 = 2$ as follows:

Let I= largest integer such that $I^2 < 2$
That is,

$$I^2 < 2 < (I+1)^2 \qquad \text{(Clearly } I = 1.\text{)}$$

Now let d_1 be the largest integer such that $\left(I + \frac{d_1}{10}\right)^2 < 2$
That is,

$$\left(I + \frac{d_1}{10}\right)^2 < 2 < \left(I + \frac{d_1 + 1}{10}\right)^2$$

It follows that $0 \leq d_1 \leq 9$ since

$$\left(I + \frac{0}{10}\right)^2 = I^2 < 2 < (I+1)^2 = \left(I + \frac{10}{10}\right)^2$$

(It easy to show that $d_1 = 4$.)
Now let d_2 be the largest integer such that $\left(I + \frac{d_1}{10} + \frac{d_2}{10^2}\right)^2 < 2$

That is

$$\left(I + \frac{d_1}{10} + \frac{d_2}{10^2}\right)^2 < 2 < \left(I + \frac{d_1}{10} + \frac{d_2 + 1}{10^2}\right)^2$$

Again it follows that $0 \leq d_2 \leq 9$.

Continuing like this we define an infinite decimal $I + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} \cdots$ with

$$\left(I + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} \cdots + \frac{d_n}{10^n}\right)^2 < 2 < \left(I + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} \cdots + \frac{d_n + 1}{10^n}\right)^2$$
$$= \left(I + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} \cdots + \frac{d_n}{10^n} + \frac{1}{10^n}\right)^2$$

If we let $S_n = I + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} \cdots + \frac{d_n}{10^n}$ then we have

$$(S_n)^2 < 2 < \left(S_n + \frac{1}{10^n}\right)^2.$$

The Completeness Axiom guarantees that $\{S_n\}$ converges to some real number L. That is, $\lim_{n\to\infty} S_n = L$ for some $L \in \mathbb{R}$.

From the properties of limits it follows that

$$\lim_{n\to\infty} S_n^2 = \lim_{n\to\infty} S_n \lim_{n\to\infty} S_n = L^2;$$
$$\lim_{n\to\infty}\left(S_n + \frac{1}{10^n}\right) = \lim_{n\to\infty} S_n + \lim_{n\to\infty} \frac{1}{10^n} = L + 0 = L$$
$$\Rightarrow \lim_{n\to\infty}\left(S_n + \frac{1}{10^n}\right)^2 = L^2.$$

and so

$$(S_n)^2 < 2 < \left(S_n + \frac{1}{10^n}\right)^2$$

$$\Rightarrow \lim_{n\to\infty} (S_n)^2 \leq 2 \leq \lim_{n\to\infty}\left(S_n + \frac{1}{10^n}\right)^2.$$

$$\Rightarrow L^2 \leq 2 \leq L^2$$

which means that $L^2 = 2$.

0.1 Finite Limits

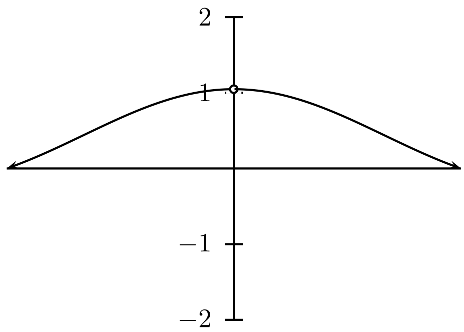
Consider the functions $f(x) = \dfrac{\sin(x)}{x}, g(x) = \dfrac{x^2-4}{x-2}$. These are examples of functions which are not defined at a particular point but yet become arbitrarily close to a particular value as x becomes close to the point at which they are not defined.

Look at the following table of values where we see that $\dfrac{\sin(x)}{x}$ becomes close 1 as x approaches 0:

	x	$\dfrac{\sin(x)}{x}$	
x	0.07	0.999183533	$\dfrac{\sin(x)}{x}$
	0.06	0.999400108	
	0.05	0.999583385	
$\downarrow$	0.04	0.999733355	$\downarrow$
	0.03	0.999850007	
	0.02	0.999933335	
	0.01	0.999983333	
0	0	not defined	1
	-0.01	0.999983333	
	-0.02	0.999933335	
	-0.03	0.999850007	
$\uparrow$	-0.04	0.999733355	$\uparrow$
	-0.05	0.999583385	
	-0.06	0.999400108	
x	-0.07	0.999183533	$\dfrac{\sin(x)}{x}$

The following is a sketch of the graph of $\dfrac{\sin(x)}{x}$. Note that there is a dot missing at the point $(0, 1)$ because $\dfrac{\sin(x)}{x}$ is not defined when $x = 0$. Nevertheless we see that $\dfrac{\sin(x)}{x}$ becomes *arbitrarily close to* 1 as x approaches 0.
We express this in writing as

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$



$$y = \frac{\sin(x)}{x}$$

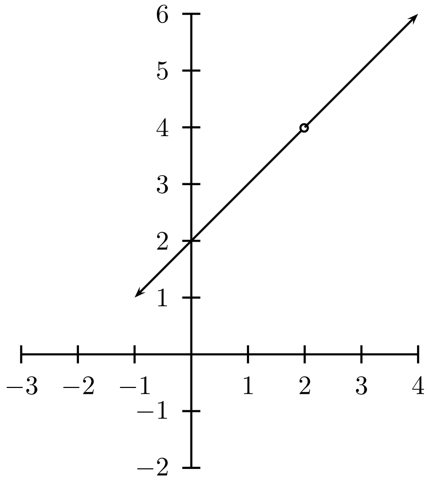
Look at the following table of values where we see that $\frac{x^2-4}{x-2}$ becomes close 4 as x approaches 2:

	x	$\frac{x^2-4}{x-2}$	
	$\parallel$	$\parallel$	
x	1.93	3.93	$\frac{x^2-4}{x-2}$
	1.94	3.94	
$\downarrow$	1.95	3.95	$\downarrow$
	1.96	3.96	
	1.97	3.97	
	1.98	3.98	
	1.99	3.99	
2	2	not defined	4
	2.01	4.01	
	2.02	4.02	
$\uparrow$	2.03	4.03	$\uparrow$
	2.04	4.04	
	2.05	4.05	
	2.06	4.06	
x	2.07	4.07	$\frac{x^2-4}{x-2}$

The following is a sketch of the graph of $\frac{x^2-4}{x-2}$. Note that there is a dot missing at the point $(2,4)$ because $\frac{x^2-4}{x-2}$ is not defined when $x=2$. Nevertheless we see that $\frac{x^2-4}{x-2}$ becomes *arbitrarily close to* 4 as x approaches 2.

We express this in writing as

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=4$$



$$y=\frac{x^2-4}{x-2}$$

Definition 0.1

Let f be a function defined over an open interval that contains a except possibly at a itself then $\lim_{x\to a} f(x)=L$ if for each $\epsilon>0$ there is $\delta>0$ such that

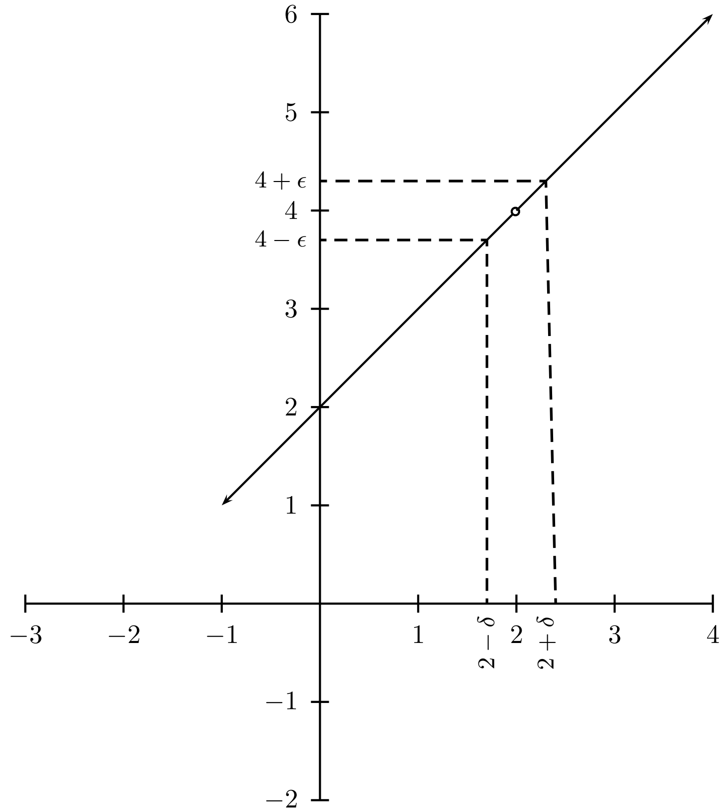
$$|f(x)-L|<\epsilon,\ \text{whenever } 0<|x-a|<\delta.$$

Informally this definition says that $f(x)$ becomes *arbitrarily close* to L

(that is, $|f(x)-L|<\epsilon$, for any $\epsilon>0$ however small)

as x becomes close enough to a

(that is, when $0<|x-a|<\delta$ for some $\delta>0$.)



Example 0.2

(i) $\lim_{x \to a} x = a$ because, for each $\epsilon > 0$, $|x - a| < \epsilon$ when $|x - a| < \epsilon$. That is, $\epsilon = \delta$.

(ii) A statement such as $\lim_{x \to a} 3 = 3$ is referring to the limit of the constant function $f(x) = 3$ as x approaches a.
In general we write $\lim_{x \to a} k = k$ to refer to the limit of the constant function $f(x) = k$ as x approaches a.

$\lim_{x \to k} k = k$ because, for each $\epsilon > 0$, $|k - k| = 0 < \epsilon$ for every $x \in \mathbb{R}$. That is, δ can be any positive real number.

(iii) Prove that $\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$:
$\left|\frac{x^2-4}{x-2} - 4\right| = \left|\frac{x^2-4-4(x-2)}{x-2}\right| = |x-2|.$
And so if we let $\epsilon = \delta$ we have :

$$\left|\frac{x^2-4}{x-2} - 4\right| < \epsilon \text{ when } |x-2| < \delta.$$

(iv) Prove that $\lim_{x\to 4} 2x+3 = 11$:

$$|2x+3-11| = |2x-8| = 2|x-4| < \epsilon \text{ when} 0 < |x-4| < \frac{\epsilon}{2}.$$

That is, $\delta = \frac{\epsilon}{2}$.

We can carefully establish simple limits such as $\lim_{x\to a} x = a$ and $\lim_{x\to a} k = k$ using the definition as follows:

Properties of limits:
If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$ then :

(i) $\lim_{x\to a} (f(x)+g(x)) = L_1 + L_2$.

(ii) $\lim_{x\to a} (f(x)g(x)) = L_1 L_2$.

(iii) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$ if $L_2 \neq 0$.

(iv) Let f and g be functions defined over an open interval I containing a with $f(x) \leq g(x) \quad \forall x \in I$.
If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to x} f(x) = L_2$ then $L_1 \leq L_2$.

There is also a version of the *Squeezing Theorem* for finite limits:

Theorem 0.3 (The Squeezing Theorem)

Let f and g and h be functions defined over an open interval I containing a.
Let $f(x) \leq g(x) \leq h(x), \quad \forall x \in I \setminus \{a\}$.
If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} g(x) = L$.

Example 0.4

Use the Squeezing Theorem to prove that $\lim_{x \to 0} x^2 \sin(x) = 0$.

$$-1 \leq \sin(x) \leq 1 \Rightarrow -x^2 \leq x \sin(x) \leq x^2$$

and since $\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0$ it follows from the Squeezing Theorem that $\lim_{x \to 0} x^2 \sin(x) = 0$.

0.1.0.2 Left-hand and right-hand limits.

Definition 0.5

$\lim_{x \to a^+} f(x) = L$ if for each $\epsilon > 0$ there is $\delta > 0$ such that

$$|f(x) - L| < \epsilon, \ \forall x \text{ whenever } 0 < x - a < \delta.$$

L is said to be the *right-hand limit* of f as x approaches a.

Definition 0.6

$\lim_{x \to a^-} f(x) = L$ if for each $\epsilon > 0$ there is $\delta > 0$ such that

$$|f(x) - L| < \epsilon, \ \forall x \text{ whenever } 0 < a - x < \delta.$$

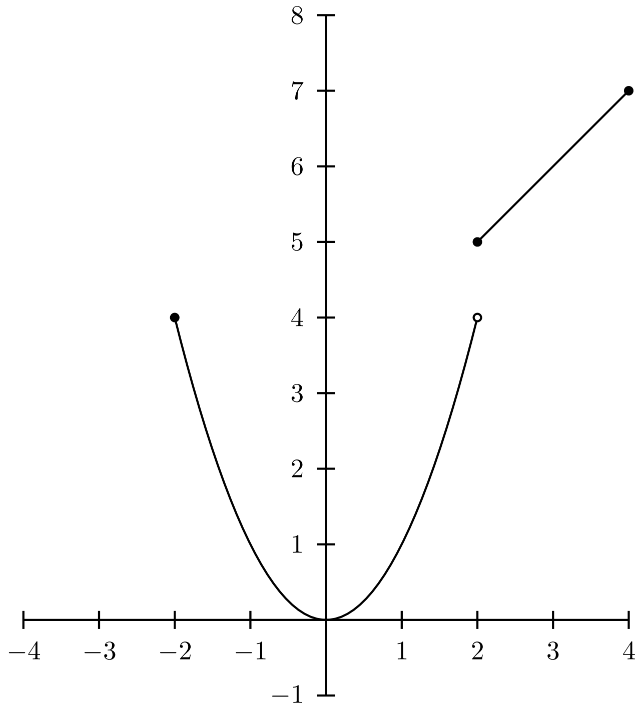
L is said to be the *left-hand limit* of f as x approaches a.

It is easy to show that if $\lim_{x \to a} f(x)$ exists and equals L, then the left-hand and right-hand limits both exist and are both equal to L and, conversely, if the left-hand and right-hand limits both exist and both have the same value L then $\lim_{x \to a} f(x)$ exists and equals L.

Example 0.7

(i) Consider the function

$$f(x) = \begin{cases} x^2, & x \in [-2, 2) \\ x + 3, & x \in [2, 4]. \end{cases}$$



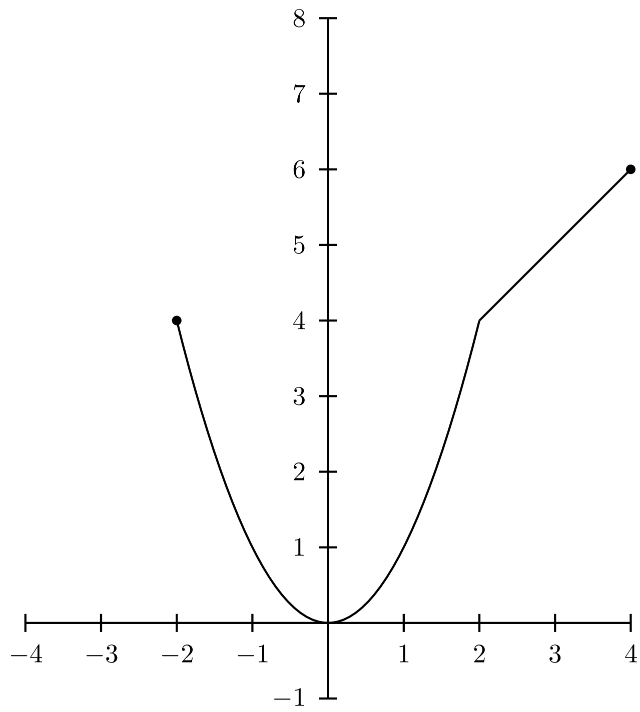
We can see that

$$\lim_{x\to 2^-} f(x) = 4 \text{ and } \lim_{x\to 2^+} f(x) = 5$$

Since $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$ we conclude that $\lim_{x\to 2} f(x)$ does not exist.

(ii) Consider the function

$$f(x) = \begin{cases} x^2, & x \in [-2, 2) \\ x + 2, & x \in [2, 4]. \end{cases}$$



We can see that

$$\lim_{x\to 2^-} f(x) = 4 \text{ and } \lim_{x\to 2^+} f(x) = 4$$

Since $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 4$ we conclude that $\lim_{x\to 2} f(x)$ exists and $\lim_{x\to 2} f(x) = 4$.